1 First-order linear partial differential equations

Theorem 1. *Let X be an open set in $\mathbb{R}^n$, $f\colon X \to \mathbb{R}$ be a $\mathbf{C}^1$ function, and $v\colon X \to \mathbb{R}^n$ be a $\mathbf{C}^1$ vector field.*

Then f satisfies the first-order linear partial differential equation:

$$\mathrm{D}\, f(x) \cdot v(x) = 0\,, \quad x \in U$$

if and only if for every solution curve γ satisfying the characteristic equation

$$\dot{x} = v(x)$$

the function f is constant on γ; that is, $\mathrm{D}(f \circ \gamma) = 0$.

Proof. Suppose f satisfies the PDE, and γ be a solution to the characteristic equation. Then by the chain rule,

$$\mathrm{D}(f \circ \gamma)(t) = \mathrm{D}\, f\bigl(\gamma(t)\bigr) \cdot \dot{\gamma}(t) = \mathrm{D}\, f(x) \cdot v(x) = 0\,, \quad x = \gamma(t) \tag{1}$$

so f is constant on the curve γ.

Conversely, suppose f is constant on every characteristic curve γ. Given $x_0 \in X$, by the existence theorem for ODEs, there exists a particular γ for which $\gamma(0) = x_0$. Then by the same equation (1), we have $\mathrm{D}\, f(x_0) \cdot v(x_0) = 0$. Since this is true for all $x_0 \in X$, f satisfies the PDE. □

Theorem 2. *Let $X \subseteq \mathbb{R}^n$ be open, and v be a $\mathbf{C}^1$ vector field. Let $x_0 \in X$ such that $v(x_0) \neq 0$ be given. Then there exists an open set $U \subseteq X$ containing x_0, and a function $k\colon U \to \mathbb{R}^{n-1}$, such that every $\mathbf{C}^1$ solution of the linear partial differential equation*

$$\mathrm{D}\, f(x) \cdot v(x) = 0\,, \quad x \in U$$

can be written in the form

$$f(x) = h(k(x))\,, \quad x \in U$$

for an arbitrary $\mathbf{C}^1$ function $h\colon \mathbb{R}^{n-1} \to \mathbb{R}$.

Proof. By ODE theory, the vector field v can be rectified locally: that is, there exists an open (connected) set U containing the given x_0 and an associated $\mathbf{C}^1$ diffeomorphism $g\colon U \to g(U)$, such that

$$\mathrm{D}\, g(x) \cdot v(x) = e_1 = (1, 0, \ldots, 0)\,, \quad \text{for all } x \in U.$$

Suppose that f satisfies the PDE. Letting $y = g(x)$, we have

$$\begin{aligned} 0 = \mathrm{D}\, f(x) \cdot v(x) &= \mathrm{D}\, f(x) \cdot \bigl([\mathrm{D}\, g(x)]^{-1} \cdot e_1\bigr) \\ &= \mathrm{D}\, f\bigl(g^{-1}(y)\bigr) \cdot \mathrm{D}\, g^{-1}(y) \cdot e_1 \\ &= \mathrm{D}(f \circ g^{-1})(y) \cdot e_1 \\ &= \mathrm{D}_1(f \circ g^{-1})(y) \end{aligned}$$

In other words, $f \circ g^{-1}$ is a $\mathbf{C}^1$ function h of all the coordinates except the first. Thus

$$(f \circ g^{-1})(y) = f(x) = h(y_2, \ldots, y_n) = h(g_2(x), \ldots, g_n(x))\,.$$

Letting $k(x) = (g_2(x), \ldots, g_n(x))$, we have the result. □

Example 1. *Solve the partial differential equation:*

$$0 = -y\,\frac{\partial f}{\partial x} + x\,\frac{\partial f}{\partial y} = \mathrm{D}\,f(x,y) \cdot v(x,y)\,, \quad v(x,y) = -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}\,.$$

Solution. The vector field v describes characteristic curves that are concentric circles centered at the origin of $\mathbb{R}^2$. Changing to polar coodinates, $x = r\cos\theta$ and $y = r\sin\theta$, rectifies the vector field:

$$\begin{aligned}\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\,\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\,\frac{\partial}{\partial y} &= -r\sin\theta\,\frac{\partial}{\partial x} + r\cos\theta\,\frac{\partial}{\partial y} \\ &= -y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y}\,.\end{aligned}$$

So the original partial differential equation is equivalent to:

$$\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) f = \frac{\partial f}{\partial\theta} = 0\,.$$

In other words, a $\mathbf{C}^1$ solution f must be constant on the circles, and depends on the radius of the point only: $f(x,y) = h(r^2) = h(x^2 + y^2)$, for any $\mathbf{C}^1$ function h. □

> The essence of the connection between a partial differential equation and a characteristic equation is that, the motion of a solid medium can be described using both the ordinary differential equations of motion of its particles, and the partial differential equations for a field.
>
> *V. I. Arnol'd.*

2 First-order quasi-linear partial differential equations

A more complicated method than the above, but based on similar principles, can be used to solve first-order quasi-linear partial differential equations.

Let x denote a point of an open set X in $\mathbb{R}^n$.

Given a partial differential equation of the following form:

$$\mathrm{D}\,f(x) \cdot a(x, f(x)) = b(x, f(x))\,, \quad a \neq 0\,, \tag{2}$$

possibly with prescribed initial values on a hypersurface in $\mathbb{R}^n$, we give a theoretical procedure to find the solution f locally.

From equation (2), we form a direction field in $X \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$ with the following vector components:

$$\sum_{i=1}^{n} a_k(x,u)\,\frac{\partial}{\partial x_i} + b(x,u)\,\frac{\partial}{\partial u}\,. \tag{3}$$

Theorem 3. *A function f is a solution to equation* (2) *if and only if its graph is an integral (hyper)surface to the direction field* (3)*.*

Proof. If f is a solution, let $\alpha(x) = (x,u) = (x, f(x))$ parameterize its graph. We claim that the image of $\mathrm{D}\,\alpha(x)$ includes $(a(x,u), b(x,u))$. This can be readily seen in the matrix equation:

$$\mathrm{D}\,\alpha(x)\cdot\xi = \begin{bmatrix} I \\ \mathrm{D}\,f(x)\end{bmatrix}\cdot\xi = \begin{bmatrix} a(x,u) \\ b(x,u)\end{bmatrix},$$

with $\xi = a(x,u)$. Thus the graph of f is an integral surface.

Conversely, if the graph of f is an integral surface, that means the above equation, for each x, holds for some vector ξ which is unknown a priori. But we can immediately derive that $\xi = a(x,u)$, and consequently the $(n+1)$th component in the matrix equation is exactly the PDE (2). □

Similar to solving other types of differential equations, we can approach the first-order quasi-linear PDE by changing variables — that is, performing diffeomorphisms to simplify the equations. To this end, we study how a diffeomorphism acts on the PDE (2) and its characteristic direction field (3).

Let $g\colon X \times \mathbb{R} \to Y \times \mathbb{R}$ be a diffeomorphism.

Applying g to the characteristic direction field (3), we obtain another direction field in $Y \times \mathbb{R}$, with components:

$$\mathrm{D}\,g(x)\cdot\big(a(x,u), b(x,u)\big)\,, \quad (x,u) = g^{-1}(y,v)\,.$$

To this direction field, there is a corresponding PDE on Y.

Then, according to Theorem 3, there is a one-to-one correspondence between the local solutions to the PDEs on the two spaces. In detail:

1. If a diffeomorphism is applied to a directional field and a hypersurface that is an integral surface to that directional field, then the image surface is automatically an integral surface to the image directional field.

2. If a hypersurface is a graph of a function, then *under certain conditions on the diffeomorphism g*, the image hypersurface will also be, at least locally, a graph of a function.

We now explain the conditions required on g. Given a particular point $(x_0, u_0) \in X \times \mathbb{R}$, the centre point for the local solution, we require:

$$\mathrm{D}\,g(x_0,u_0)\cdot e_{n+1} = e_{n+1}\,, \quad \text{that is,}\ \left.\frac{\partial g}{\partial u}\right|_{(x_0,u_0)} = e_{n+1}\,. \tag{4}$$

Suppose that a hypersurface in Y is parameterized by $(y, v) = (y, h(y))$. In implicit form, the surface is described by:

$$0 = H(y, v) = v - h(y)\,.$$

Then, the corresponding hypersurface in X has the following implicit form:

$$0 = F(x, u) = H(g(x, u))\,.$$

This hypersurface can be expressed as a graph $(x, u) = (x, f(x))$ near (x_0, u_0) provided that $\partial F/\partial u \neq 0$ at that point. But:

$$\frac{\partial F}{\partial u} = \left\langle \nabla H, \frac{\partial g}{\partial u} \right\rangle = 1\,, \quad \text{at } (x, u) = (x_0, u_0),$$

by condition (4).

Analogously, any hypersurface that is a graph in X-space can be expressed as a graph near $g(x_0, u_0)$ in Y-space also.

To solve the PDE (2), we begin by selecting a diffeomorphism $\tilde{g}$ that rectifies the direction field (3), to the extremely simple one:

$$\frac{\partial}{\partial y_1}\,, \tag{5}$$

corresponding to the PDE

$$\frac{\partial h}{\partial y_1} = 0\,. \tag{6}$$

Although $\tilde{g}$ does not necessarily satisfy condition (4), we can make a diffeomorphism that does so by composing $\tilde{g}$ with an invertible linear transformation M. Let $g = M \circ \tilde{g}$ where M fixes e_1 and moves $\mathrm{D}\,\tilde{g}(x_0, u_0) \cdot e_{n+1}$ to e_{n+1}. The two conditions are never contradictory — that is, e_1 and $\mathrm{D}\,\tilde{g}(x_0, u_0) \cdot e_{n+1}$ are always linearly independent — for $\mathrm{D}\,\tilde{g}$ is one-to-one, and brings some vector (a, b) to e_1, with $a \neq 0$, so $\mathrm{D}\,\tilde{g}$ cannot possibly move e_{n+1} to a multiple of e_1 too.

Evidently g also rectifies the field (3) to the field (5).

The solution to the simplified PDE (6) is clearly $h(y_1, \dots, y_n) = k(y_2, \dots, y_n)$ for an arbitrary $\mathbf{C}^1$ function $k\colon \mathbb{R}^{n-1} \to \mathbb{R}$. (Assume the domain Y is connected.) Resubstituting, we find that the solution f to the PDE (2) near (x_0, u_0) is implicitly given by this equation:

$$0 = g_{n+1}(x, u) - k\big(g_2(x, u), \dots, g_n(x, u)\big)\,, \quad u = f(x)\,.$$

Formally, we have proved the following:

Theorem 4. *The first-order quasi-linear partial differential equation, via a change of variables, is locally equivalent to another differential equation of the form $\partial h/\partial y_1 = 0$, and is solvable locally.*

Example 2. *We solve Euler's PDE for a homogeneous function:*

$$\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} = \alpha u\,, \quad \alpha \neq 0\,,$$

with the specified boundary value:

$$u(x_1, \ldots, x_{n-1}, 1) = h(x_1, \ldots, x_{n-1})\,.$$

Solution. The characteristic direction field,

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \alpha u \frac{\partial}{\partial u}\,,$$

has a nice geometric structure in its x-component. The x-component consists of rays emanating from the origin; the characteristic curves are lines through the origin. This observation suggests the following change of variable:

$$y_i = \frac{x_i}{x_n}\,,\ \text{for } i = 1, \ldots, n-1\,; \quad y_n = x_n\,, \quad v = u\,.$$

(We choose to divide by x_n rather than the other variables x_i because $x_n = 1$ plays a special role in the boundary condition.)

The position of a point along a characteristic curve is described by the value of y_n alone, while the different variables $y_1, \ldots, y_{n-1}$ distinguish between the different characteristic curves. This scheme necessarily makes the partial derivatives transversal to the characteristics, those with respect to $y_1, \ldots, y_{n-1}$, vanish. The characteristic direction field in the new coordinate system becomes:

$$y_n \frac{\partial}{\partial y_n} + \alpha v\,.$$

This direction field is not the simplest rectification possible, but the diffeomorphism (change of variables) is easy to work with. The corresponding PDE in the new variables is:

$$\frac{\partial v}{\partial y_n} = \frac{\alpha v}{y_n}\,,$$

whose solution is obtained by single-dimensional integration:

$$v(y) = y_n^{\alpha}\, k(y_1, \ldots, y_{n-1})\,,$$

for some function k.

Re-expressing the solution in the variables x_i, we find the general solution to the original PDE is

$$u(x) = u = v = v(y) = x_n^{\alpha}\, k\left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right)\,.$$

Finally, the boundary condition asserts that

$$h(x_1, \ldots, x_{n-1}) = u(x_1, \ldots, x_{n-1}, 1) = k(x_1, \ldots, x_{n-1})\,,$$

and thus:

$$u(x_1, \ldots, x_n) = x_n^\alpha \, h\left(\frac{x_1}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right) .$$

(The function u satisfies $u(\lambda x) = \lambda^\alpha u(x)$ for $\lambda > 0$.) □

Remark. The solution obtained is not defined for $x_n \leq 0$. This is due to a defect of the coordinate system (y, v) we had chosen: to form it, we had to divide by x_n, so clearly that cannot be zero. (At the boundary of the domain, $x_n = 0$.) We may also observe that the original PDE has a "singularity" at $x = 0$; the characteristic direction field is vertical (has only no x component) there.

However, it may still be possible (or may not be) to put together a solution that is also defined for some $x_n \leq 0$, $x \neq 0$, if additional boundary conditions are provided. □

Example 3. *Solve the following initial-value problem:*

$$xu \, \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0 \,, \quad u(x, 0) = x \,.$$

Solution. First, note that the characteristic direction field describes no change in u. So u can be held constant, and looking at the x- and y- components alone, the characteristic direction field looks the same as the direction field for the following ODE:

$$\frac{dy}{dx} = -\frac{1}{ux} \,.$$

This ODE can be solved explicitly to give the characteristic curves:

$$y = -\frac{1}{u} \log(sx) \,, \quad \text{or} \quad e^{-yu} = sx \,,$$

for a parameter or constant of integration s. From this description, we derive a coordinate system that rectifies the original characteristic direction field:

$$s = \frac{e^{-yu}}{x} \,, \quad t = x \,, \quad v = u \,.$$

Indeed, we can verify that:

$$\begin{aligned} \frac{\partial}{\partial t} = \frac{\partial x}{\partial t} \frac{\partial}{\partial x} + \frac{\partial y}{\partial t} \frac{\partial}{\partial y} + \frac{\partial u}{\partial t} \frac{\partial}{\partial u} &= \frac{\partial}{\partial x} - \frac{1}{vt} \frac{\partial}{\partial y} \\ &= \frac{1}{xu} \left(xu \, \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) , \end{aligned}$$

and so the PDE in the new coordinates becomes the simple $\partial v / \partial t = 0$, whose solution is:

$$u(x, y) = u = v = v(s, t) = k(s) = k\left(\frac{e^{-yu}}{x}\right) ,$$

for some function k. The prescribed boundary value for u tells us that

$$x = u(x, 0) = k\left(\frac{1}{x}\right),$$

and thus we find that $u = u(x, y)$ can be described implicitly as

$$x = ue^{-yu}. \qquad \square$$

Remark. Usually the solution to $dy/dx = -1/ux$ is written:

$$y = -\frac{1}{u}\log|x| + c.$$

for some constant c instead of s above. The problem with this expression is not only that the absolute value is difficult to work with, but also the equation actually describes a set with two connected components, one with $x < 0$ and the other with $x > 0$. So there are really two separate characteristic curves with the same parameter c.

In some cases, blithely ignoring this aspect of the coordinate system may lead to the wrong answer. For example, consider the following linear PDE, with the change of variable indicated:

$$0 = x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = t\frac{\partial u}{\partial t}, \quad s = xy, t = x.$$

It is *not* the case that every solution to this PDE is a function of $s = xy$. (The equation $s = xy$ describes two characteristic curves. There is no contradiction with our previous results, which state that the solution is only *locally* a function of s. More specifically, the equation $\partial u/\partial t = 0$ does not mean that u is constant on the set $\{s_0 = s = xy\}$; it is only constant on *each connected piece of that set.*) $\square$

3 Existence of solutions to the Cauchy problem

The previous examples were simple enough that, we were able to derive the general solution to the PDE, and substitute in the initial values only after changing back to the (x, u) variables from the (y, v) variables.

One may want to take into account the initial values earlier when changing to the (y, u) variables. This is possible if the initial hypersurface — an $(n-1)$-dimensional manifold in x-space — is transversal to each characteristic curve, projected from (x, u)-space onto x-space, passing through each of its points. In that case, whenever we obtain a diffeomorphism $g\colon (x, u) \mapsto (y, v)$ such that the PDE reduces to $\partial v/\partial y_n = 0$, we can, theoretically, prepare another diffeomorphism $\tilde{g}\colon y \mapsto z$, where the initial hypersurface in (z, v)-space is locally parameterized by $(z_1, \dots, z_{n-1}, 0, v)$, while the PDE expressed in the variables (z, v) retains the rectified form $\partial v/\partial z_n = 0$. Then the solution to the PDE, $v(z) = k(z_1, \dots, z_{n-1})$, will be already directly expressed in terms of the given initial values, albeit in a different coordinate system.

The above reasoning proves:

Theorem 5. *The Cauchy problem — that is, finding the solution of a PDE with prescribed initial values — has a unique solution in a neighbourhood of each point of the initial hypersurface, provided the transversal property holds.*

To be concrete about the transversal property, suppose the initial $(n-1)$-dimensional hypersurface in x-space is represented by a coordinate chart $x = \phi(\xi)$ for $\xi \in \mathbb{R}^{n-1}$, and the initial values there are given by $u(x) = h(x)$. The tangent line to the characteristic curves, projected onto x-space, is represented by the vector $a(x, u)$ in the PDE (2). Then the hypersurface and the characteristic curves' projections are transversal whenever

$$\det \begin{bmatrix} \mathrm{D}\,\phi(\xi) & a(x,u) \end{bmatrix} \neq 0\,, \quad x = \phi(\xi)\,, u = h(x)\,,$$

for the above determinant is zero if and only if the vector $a(x, u)$ lies in the tangent space of the hypersurface at the point x.

If the transversal property fails to hold, clearly the Cauchy problem may not be solvable or may not have a unique solution. For example, if the initial hypersurface coincides with a characteristic curve, the function h might simply assign the "wrong" values to u on that characteristic curve, that are incompatible with the PDE itself.

References

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[2] Fritz John. *Partial Differential Equations*, fourth ed. Springer-Verlag, 1986.